Math 601 Midterm 2

Name: ____________________

This exam has 8 questions, for a total of 100 points.

Please answer each question in the space provided. You need to write **full solutions**. Answers without justification will not be graded. Cross out anything the grader should ignore and circle or box the final answer.

Question	Points	Score
1	15	
2	10	
3	15	
4	15	
5	15	
6	10	
7	10	
8	10	
Total:	100	

Question 1. (15 pts)

Determine whether each of the following statements is true or false. You do NOT need to explain.

(a) If A is a square matrix with all entries being positive, then $\det(A)$ is positive.

(b) An $(n \times n)$ matrix is invertible if and only if its n column vectors form a linearly independent set.

(c) Let U and V be subspaces of $\mathbb{R}^{10}$. If $\dim U = 7$ and $\dim V = 3$, then $U + V = \mathbb{R}^{10}$.

(d) Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$, then W is a subspace of $\mathbb{R}^3$.

(e) Let A be a 3×3 matrix. If A is diagonalizable, then A has 3 distinct eigenvalues.

Solution:

(a) False

(b) True

(c) False

(d) True

(e) False

Question 2. (10 pts)

Find all eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 4-3i \\ 4+3i & 3 \end{bmatrix}$$

Solution:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 4-3i \\ 4+3i & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - (4-3i)(4+3i) = \lambda^2 - 6\lambda - 16$$

When $\lambda = 8$, the eigenvector is

$$v = \begin{bmatrix} 4-3i \\ 5 \end{bmatrix}$$

When $\lambda = -2$, the eigenvector is

$$w = \begin{bmatrix} 4-3i \\ -5 \end{bmatrix}$$

Question 3. (15 pts)

The eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ are $\lambda_1 = 1$ with $v_1 = (1, 0, 0)^T$, $\lambda_2 = 2$ with $v_2 = (1, 1, 0)^T$ and $\lambda_3 = 3$ with $v_3 = (1, 2, 1)^T$.

(a) Find the general solution to the system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

Solution: The general solution is

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

In other words,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2 e^{2t} + c_3 e^{3t} \\ c_2 e^{2t} + 2c_3 e^{3t} \\ c_3 e^{3t} \end{bmatrix}$$

(b) Find a specific solution $\mathbf{x}(t)$ such that

$$\mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

when $t = 0$.

Solution:

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + c_3 \\ c_1 + 2c_3 \\ c_3 \end{bmatrix}.$$

Therefore, we need to solve for c_1, c_2 and c_3 of the following linear system

$$\begin{bmatrix} c_1 + c_2 + c_3 \\ c_1 + 2c_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

which has a unique solution $c_1 = 3, c_2 = 2$ and $c_3 = 1$. So the solution satisfying the given initial condition is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 3e^t + 2e^{2t} + e^{3t} \\ 2e^{2t} + 2e^{3t} \\ e^{3t} \end{bmatrix}$$

Question 4. (15 pts)

F is linear transformation from $\mathbb{P}_2(t)$ to $\mathbb{P}_2(t)$ defined by

$$F(a+bt+ct^2) = (a+2b-c) + (b+c)t + (a+b-2c)t^2.$$

Recall that $S = \{1, t, t^2\}$ is a basis of $\mathbb{P}_2(t)$.

(a) Write down the matrix representation of F relative to the basis $S = \{1, t, t^2\}$.

Solution: Notice that

$$F(1) = 1 + t^2$$
$$F(t) = 2 + t + t^2$$
$$F(t^2) = -1 + t - 2t^2$$

$$[F]_S = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

(b) Find the kernel of F.

Solution: First reduce the matrix $[F]_S$ in part (a) to its echelon form, which is

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\mathrm{Ker}F = \mathrm{span}\{\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}\}$. In other words, $\mathrm{Ker}F$ is spanned of one polynomial $3 - t + t^2$.

(c) Find the dimension of the image of F.

Solution:

$$\dim(\mathrm{Im}F) + \dim(\mathrm{Ker}F) = 3$$

From part (b), we know that $\dim(\mathrm{Ker}F) = 1$. So $\dim(\mathrm{Im}F) = 2$.

(d) Is F is an isomorphism? Explain.

Solution: Since $\mathrm{Ker}F$ is not equal to the zero vector space $\{0\}$, we see that F is not an isomorphism.

Question 5. (15 pts)

Let U be the subspace of $\mathbb{R}^4$ spanned by

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 8 \\ 5 \\ 1 \\ 4 \end{bmatrix}.$$

(a) Find an orthonormal basis of U.

Solution:

$$w_1 = v_1 = (0, 1, 0, 0)$$

$$w_2 = v_2 - \frac{\langle w_1, w_2 \rangle}{\langle w_1, w_1 \rangle} w_1 = (2, 0, 0, 1)$$

$$w_3 = v_3 - \frac{\langle w_1, w_3 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle w_2, w_3 \rangle}{\langle w_2, w_2 \rangle} w_2 = (0, 0, 1, 0)$$

So

$$u_1 = \frac{w_1}{\|w_1\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \frac{w_3}{\|w_3\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

(b) Find the projection of

$$w = \begin{bmatrix} 4 \\ 2 \\ 5 \\ 0 \end{bmatrix}$$

onto U.

Solution:

$$\begin{aligned} \text{Proj}_U(w) &= \langle w, u_1 \rangle u_1 + \langle w, u_2 \rangle u_2 + \langle w, u_3 \rangle u_3 \\ &= 2u_1 + \frac{8}{\sqrt{5}} u_2 + 5u_3 \\ &= (\frac{16}{5}, 2, 5, \frac{8}{5})^T \end{aligned}$$

Question 6. (10 pts)

Determine whether the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is diagonalizable.

Solution: Use cofactor expansion along the first column

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = (1-\lambda)\lambda^2$$

So $\lambda = 1$ and 0.

When $\lambda = 1$, solve for the kernel of the matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

We find an eigenvector $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

When $\lambda = 0$, solve for the kernel of the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We find an eigenvector $w_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $w_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

We can verify that v, w_1 and w_2 are linearly independent. It follows that A has 3 linearly independent eigenvectors, so A is diagonalizable.

Question 7. (10 pts)

Let V be the vector space spanned by the basis $S = \{e^x, xe^x, e^{-x}\}$. Determine whether the functions

$$g_1(x) = e^x + xe^x + e^{-x}$$

$$g_2(x) = 2e^x + 3xe^x + 4e^{-x}$$

$$g_3(x) = xe^x + 5e^{-x}$$

are linearly independent or not.

Solution:

$$[g_1]_S = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$[g_2]_S = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$[g_3]_s = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$$

Consider the matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 1 & 4 & 5 \end{bmatrix}$$

its echelon form is

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

which has rank 3. Therefore g_1, g_2 and g_3 are linearly independent.

Question 8. (10 pts)

Let V be the vector space spanned by $\{\sin x, \cos x, \sin(2x), \cos(2x)\}$. Accept as a fact that

$$S = \{\sin x, \cos x, \sin(2x), \cos(2x)\}$$

form a basis for V. Let

$$T(f) = f - 2f'$$

be a linear transformation from V to V. Determine whether T is an isomorphism, that is, whether T is invertible. (**Hint: first find the matrix representation of T with respect to S.**)

Solution:

$$T(\sin x) = \sin x - 2\cos x$$

similarly, we have

$$T(\cos x) = \cos x + 2\sin x$$

$$T(\sin(2x)) = \sin(2x) - 4\cos(2x)$$

$$T(\cos(2x)) = \cos(2x) + 4\sin(2x)$$

So

$$[T]_{\mathfrak{B}} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -4 & 1 \end{bmatrix}$$

Notice that the determinant of $[T]_{\mathfrak{B}}$ is $85 \neq 0$. (Alternatively, show that $[T]_{\mathfrak{B}}$ has rank 4.) So T is an isomorphism.